Search Theory

Renato Molina *

May 23, 2019

Contents

1 Motivation

In most general equilibrium models we assume a frictionless economy. This assumption implies that interactions between agents occur smoothly and willingly; for example, if an individual is looking for a job or a spouse, they can find matches immediately. This property implies that solving for the planners problem is equivalent to the decentralized solution. In practice, however, it is hard to justify such assumption and we have to develop the tools to analyze the interactions between agents when they do not occur smoothly. Search theory fills this void and will help us analyze how agents behave when faced with frictions. The applications of these developments will allow us to answer questions of how unemployment or crime rates are affected by the policies that rule individuals in the economy.

*Most of these notes follow the third edition of the book "Recursive Macroeconomic Theory" by L. Ljungqvist & T. Sargent. All errors are my own.

2 Utility Assumptions

We assume that individuals are infinitely lived, but risk neutral and have a preference over time that is strictly positive:

$$\int_0^\infty e^{-rt}c(t)dt \tag{2.1}$$

where workers can be either employed or unemployed, and seek to maximize their lifetime utility.

3 Offer Arrival Assumptions

The strongest assumption in the the model is that offers have some arrival rate $\lambda \in \mathbb{R}^+$ that can be either individual specific or general for the economy. We say that benefits associated with an offer have a probability distribution, with cdf $F(\bullet)$ and support $\{\underline{\text{W}}, \bar{W}\}$. While searching, individuals get some benefits, but once they accept an offer they get a constant stream of benefits w (i.e., constant wage). Agents can loose their status with a rate of destruction or separation $s \in \mathbb{R}^+$.

3.1 Distribution Assumption

Most of the time we will assume that the rate of arrival for offers follows a Poisson process, which has the nice property of begin memoryless. That is, the average rate of arrival is always the same regardless of the previous periods. Formally, these features are equivalent to say that the length of time between arrivals is independent and identically distributed (iid). We write the exponential density for time between arrivals as:

$$f(t) = \alpha e^{-\alpha t} \tag{3.1}$$

with α as the arrival rate of events. Integrating the density function renders the probability of an event occurring within some amount of time t. The associated counting process of how many offers arrived in the interval $[0, t]$ is given by:

$$N_t = \sum_{n=1}^{\infty} \mathbb{I}_{T_n \leq t} \tag{3.2}$$

We say that N_t is distributed Poisson with parameter $alpha$:

$$Pr(N_t = n) = \frac{(\alpha t)^n e^{-\alpha t}}{n!} \tag{3.3}$$

which is known as the Poisson distribution. This pmf gives us the probability that n offers arrive over some time period. Note that αt is the mean number of offers. Now we look at the probability that an offer arrives within some arbitrarily small time period:

$$Pr(N_{dt} = 0) = e^{-\alpha dt} \cong 1 - \alpha dt \tag{3.4}$$

$$Pr(N_{dt} = 1) = e^{-\alpha dt}\alpha dt \cong (1 - \alpha dt)\alpha dt \cong \alpha dt \tag{3.5}$$

$$\tag{3.6}$$

In words, we say that there is a high chance of getting zero offers if the interval is too small, unless the arrival rate is comparatively very high. On the other hand, the chances of getting one or more offers decreases accordingly as time converges to zero. This property can be interpreted as that within a small time interval a certain event occurs, or it does not occur, there are no multiple events.

4 Utility for Matched Agents

When an agent is matched, in this case the individual getting a job, her utility is given by the value function over her lifetime. If we look to her value function over a small time interval we get:

$$V_E(w) = \frac{1}{1+rdt}(wdt + sdtV_U + (1-sdt)V_E(w)) \tag{4.1}$$

In other words, her wage now, the value of being unemployed and the value of remaining unemployed. In most occasions, however, we are concerned with the reservation wage. We can derive the reservation wage through two methods, discrete and continuous time.

4.1 Discrete Time Utility

At wages above the reservation wage, the individual will be encouraged to accept job offers and will work instead of remaining unemployed. Consider the following expression:

$$U = \mathbb{E}\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{4.2}$$

with risk-neutral agents. The value of getting a job forever is $\frac{w}{1-\beta}$, while the unemployed benefits are b. The value of an offer, also referred to as wage draw, is then given by:

$$V(w) = \max\left\{\frac{w}{1-\beta}, b + \beta\int V(w')dw\right\} \tag{4.3}$$

In this simple decision model, the agent decides wether to accept the wage offer with certainty forever, or take unemployment benefits and the wage offer next period.

We are interested in the wage offer that makes the individual indifferent between having a job or being unemployed. We denote this reservation wage as w_R. The value of accepting such offer is given by$V(w_R) = \frac{w_R}{1-\beta}$. The policy function is now given by:

$$V(w) = \begin{cases} \frac{w_R}{1-\beta} = b + \beta\int_0^B V(w')dF(w'); & w \leq w_R \\ \frac{w_R}{1-\beta}; & w \geq w_R \end{cases} \tag{4.4}$$

with B as some upper bound on the cdf of $F(w)$. It follows that our indifference point is given by:

$$\frac{w_R}{1-\beta} = b + \beta\int_0^B V(w')dF(w') \tag{4.5}$$

We can work with this expression a little bit further. Consider splitting the integral as follows:

$$\frac{w_R}{1-\beta} = b + \beta\int_0^{w_R} V(w')dF(w') + \beta\int_{w_R}^B V(w')dF(w')$$

Because the integral of a pdf over it domain is equal to 1, we can multiply the right hand side (RHS) by that unit element. In our our example this looks as follows:

$$\frac{w_R}{1-\beta}\int_0^B dF(w') = b + \beta\int_0^{w_R} V(w')dF(w') + \beta\int_{w_R}^B V(w')dF(w')$$

We can now split the integral once more and insert the policy rule into the RHS:

$$\frac{w_R}{1-\beta}\left(\int_0^{w_R} dF(w') + \int_{w_R}^{B} dF(w')\right) = b + \beta\int_0^{w_R} \frac{w_R}{1-\beta} dF(w') + \beta\int_{w_R}^{B} V(w')dF(w')$$

which we can rearrange as:

$$\frac{w_R}{1-\beta}\left(\int_0^{w_R} dF(w') + \int_{w_R}^{B} dF(w')\right) - \beta\frac{w_R}{1-\beta}\int_0^{w_R} dF(w') = b + \beta\int_{w_R}^{B} V(w')dF(w')$$

Distributing terms then gives:

$$\frac{w_R}{1-\beta}\int_0^{w_R} dF(w') + \frac{w_R}{1-\beta}\int_{w_R}^{B} dF(w') - \beta\frac{w_R}{1-\beta}\int_0^{w_R} dF(w') = b + \beta\int_{w_R}^{B} V(w')dF(w')$$

Simplifying and replacing $V(w')$ for an arbitrary wage w' then gives the following expression:

$$w_R\int_0^{w_R} dF(w') = b + \beta\int_{w_R}^{B} \frac{w'}{1-\beta} dF(w') - \frac{w_R}{1-\beta}\int_{w_R}^{B} dF(w')$$

Rearranging again and merging integrals:

$$w_R\int_0^{w_R} dF(w') - b = \int_{w_R}^{B} \frac{\beta w' - w_R}{1-\beta} dF(w')$$

Adding the remaining half of the distribution multiplied by w_R:

$$w_R\int_0^{w_R} dF(w') - b + w_R\int_{w_R}^{B} dF(w') = \int_{w_R}^{B} \frac{\beta w' - w_R}{1-\beta} dF(w') + w_R\int_{w_R}^{B} dF(w')$$

Which we can finally rearrange to:

$$w_R - b = \int_{w_R}^{B} \frac{\beta w' - w_R}{1-\beta} dF(w') + \frac{1-\beta}{1-\beta} w_R\int_{w_R}^{B} dF(w')$$

$$w_R = \frac{\beta}{1-\beta}\int_{w_R}^{B} (w' - w_R)dF(w') + b \tag{4.6}$$

Equation (4.6) is known as the **fundamental equation of discrete job search**. The left hand side (LHS) is utility per period of accepting an offer of exactly w_R, while the RHS is the utility of rejecting such offer. The wage the solves this equation is the reservation wage.

5 Continuous Time Utility

When we work with continuous time we have to define the flows of people getting employed, fired and looking for a job again. Recall the individual's utility was given by:

$$U = \int_{0}^{\infty} e^{-rt} c(t) dt$$

Previously, we defined the value of being unemployed in a small interval of time as:

$$V_E(w) = \frac{1}{1+rdt}(wdt + sdtV_U + (1-sdt)V_E(w)) \tag{5.1}$$

which, in continuous time, requires an arbitrary decrease in the time interval to an infinitesimal value. To do so, we'll have to manipulate the equation a little further. Recall that L'Hopitals rule dictates:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \tag{5.2}$$

a property that applies whenever we have non-well defined limits. We want to rearrange $V_E(w)$ to be able to use this property as follows:

$$(1+rdt)V_E(w) = wdt + sdtV_U + (1-sdt)V_E(w)$$

$$rdtV_E(w) = wdt + sdtV_U - sdtV_E(w)$$

$$1 = \frac{wdt + sdtV_U - sdtV_E(w)}{rdtV_E(w)} \tag{5.3}$$

Using L'Hopital, and taking partials with respect to dt we have:

$$\lim_{dt \to 0} \frac{wdt + sdtV_U - sdtV_E(w)}{rdtV_E(w)} = \lim_{dt \to 0} \frac{w + sV_U - sV_E(w)}{rV_E(w)} = 1 \tag{5.4}$$

Now, we solve for the "Flow Bellman" equation:

$$rV_E(w) = w + sV_U - sV_E(w)$$

$$\implies V_E(w) = \frac{w + sV_U}{r+s} \tag{5.5}$$

By now, we know the reservation wage is the wage level that makes the individual indifferent between working and being unemployed. In continuous time this is equivalent to:

$$V_E(w_R) = V_U$$

$$\frac{w_R + sV_U}{r+s} = V_U$$

$$\iff w_R = rV_U \tag{5.6}$$

Therefore, to estimate the reservation wage we need the value of being unemployed. Recall that the rate of arrival was given by λ:

$$V_U = \frac{1}{1+rdt}\left(bdt + \lambda dt \int_{\underline{w}}^{\bar{w}} \max\{V_E(w'), V_u\} dF(w') + (1-\lambda dt)V_U\right) \tag{5.7}$$

The value being unemployed during a small interval of time is given by the employment benefits, plus the chance of getting an offer, and the continuation value of being unemployed. Similarly, we can solve this equation using a similar approach as with the value of begin employed:

$$(1+rdt)V_U = bdt + \lambda dt \int_{\underline{w}}^{\bar{w}} \max\{V_E(w'), V_u\} dF(w') + (1-\lambda dt)V_U$$

$$(rdt+\lambda dt)V_U = bdt + \lambda dt \int_{\underline{w}}^{\bar{w}} \max\{V_E(w'), V_u\} dF(w')$$

$$\iff 1 = \frac{bdt + \lambda dt \int_{\underline{w}}^{\bar{w}} \max\{V_E(w'), V_u\} dF(w')}{(rdt+\lambda dt)V_U}$$

We can use L'Hopital again to get:

$$\frac{b + \lambda \int_{\underline{w}}^{\bar{w}} \max\{V_E(w'), V_u\} dF(w')}{(r+\lambda)V_U} = 1$$

$$(r+\lambda)V_U = b + \lambda \int_{\underline{w}}^{\bar{w}} \max\{V_E(w'), V_u\} dF(w')$$

$$rV_U = b + \lambda \int_{\underline{w}}^{\bar{w}} \max\{V_E(w'), V_u\} dF(w') - \lambda V_U$$

Because V_u is not a function of w, we can pull it inside the integral:

$$rV_U = b + \lambda \int_{\underline{w}}^{\bar{w}} \max\{V_E(w') - V_u, 0\} dF(w') \tag{5.8}$$

To solve for the reservation wage, we substitute (5.6) into (5.8):

$$w_R = b + \lambda \int_{\underline{w}}^{\bar{w}} \max\{V_E(w') - V_u, 0\} dF(w') \tag{5.9}$$

Finally, the max operator inside the integral can take only two values, either 0 or the surplus value beyond the reservation wage. It follows that we can take (5.5) and and manipulate it as follows:

$$\begin{aligned} V_E(w) - V_u &= \frac{w + sV_U}{r+s} - V_U \\ &= \frac{w - rV_U}{r+s} \\ &= \frac{w - rw_R}{r+s} \end{aligned} \tag{5.10}$$

which we can substitute into (5.9):

$$
\begin{aligned}
w_R &= b + \lambda \int_{w_R}^{\bar{w}} \frac{w' - rw_R}{r+s} dF(w') \\
\iff w_R &= b + \lambda(1 - F(w_R))\mathbb{E}\left[\frac{w' - rw_R}{r+s}\middle| w' \geq w_R\right] \qquad (5.11)
\end{aligned}
$$

Equation (5.11) is known as the **fundamental reservation wage equation**, and it describes the point at which individuals will be indifferent between being employed or unemployed.

6 Optimal Unemployment Insurance

This section is based on the work by Hopenhayn and Nicolini, and relates to the prevalence of the unemployment insurance for agents in the economy with frictions. Previously we studied how agents make decisions about taking a job offer or not, a decision that is a function of the parameter b. b can be interpreted as the unemployment insurance administrated by governments when individuals are jobless. The idea is that the social planner seeks to minimize its cost while providing insurance, but at the same time encouraging agents to search for jobs instead of remaining unemployed.

6.1 Set-up

Agents seek to maximize their expected utility:

$$
E\sum_{t=0}^{\infty} \beta^t [u(c_t) - a_t] \qquad (6.1)
$$

The term a appears in the utility as a way to represent that agents might get negative utility based on how hard they look for jobs. We let all jobs pay $w > 0$, where there is some probability $p(a)$ of an agent getting a job offer. We assume there are no savings, and therefore the planner provides the only method of consumption without jobs.

6.2 Autarky

First, we examine the autarky case when there is no unemployment insurance and no mechanism for savings either. That is, the only way of generating consumption is by getting a job. Furthermore, there is no separation rate, which means that jobs last forever, as well as no search while employed. It follows there are only two possibles states: employed (E) and unemployed (U). Operationally, these are defined as follows:

$$
V_E = \frac{u(w)}{1-\beta} \qquad (6.2)
$$

$$
V_U = \max_a \{u(0) - a + \beta[p(a)V_E + (1 - p(a))V_U]\} \qquad (6.3)
$$

which imples:

$$
\begin{aligned}
\frac{\partial V_U}{\partial a} &= -1 + \beta[V_E - V_U]p'(a) \\
\implies \beta[V_E - V_U]p'(a) &\leq 1 \qquad (6.4)
\end{aligned}
$$

This expression holds with equality at an interior solution, $a \geq 0$. The value of a that solves the FOC is the optimal search effort in autarky.

6.3 Unemployment Insurance with Full Information

Suppose that search effort can be optimally observed, and that it can be controlled, as well as consumption. The social planner provides insurance that can make individuals better off while keeping minimum cost. Let V be the value the planner wants to provide to workers during the course of their lifetimes:

$$\begin{aligned} C(V) = & \min_{c,a,V_U} \{c + \beta[1-p(a)]C(V_U)\} \\ & \text{s.t.} \\ & V \leq u(c) - a + \beta[p(a)V_E + (1-p(a))V_U] \end{aligned} \tag{6.5}$$

In words, the cost associated with providing value V, $C(V)$, is equal to the amount given in consumption plus the discounted cost of providing the same value in the future, given that the individual is still unemployed. The constraint in this problem is called the **promise-keeping constraint**, because the planner promises agents a value that is equal to the RHS of the constraint. If the promise is kept, the constraint holds with equality.

Solving this maximization problem renders the following first order conditions (FOCs):

$$\frac{\partial C(V)}{\partial c} = 1 - \theta u'(c) \tag{6.6}$$

$$\frac{\partial C(V)}{\partial a} = -\beta p'(a)C(V_U)1 - \theta[1 - \beta p'(a)(V_E - V_U)] \tag{6.7}$$

$$\frac{\partial C(V)}{\partial V_U} = -\beta p'(a)C'(V_U) - \theta[\beta(1-p'(a))] \tag{6.8}$$

which imply:

$$\frac{1}{u'(c)} = \theta \tag{6.9}$$

$$C(V_U) = \theta\left[\frac{1}{\beta p'(a)} - (V_E - V_U)\right] \tag{6.10}$$

$$C'(V_U) = \theta \tag{6.11}$$

Applying the envelope condition:

$$\begin{aligned} \frac{\partial C(V)}{\partial V} &= \theta \\ \iff C'(V_U) &= C'(V) \end{aligned} \tag{6.12}$$

Because C is convex, we have that $V = V_U$. Hence, during unemployment, the value provided by the planner is exactly the same the as the value of unemployment. In other words, consumption and search intensity are constant. However, this is not true across states, as V_E is not necessarily equal to V_U.

6.4 Incentive Problem: Search Intensity Without Full Information

In the previous section, we saw that if the planner was able to provide a value to unemployed agents, it will give agents the same value of unemployment. However, those results rely on the perfect information assumption. If the planner is not able to observe what agents do, they will optimize according to the autarky problem, while the planner provides some level of consumption at the relevant cost. It follows that:

$$C(V_U) = \theta \left[\frac{1}{\beta p'(a)} - (V_E - V_U) \right] \tag{6.13}$$

but from the autarky problem, we also know that:

$$\beta[V_E - V_U]p'(a) \leq 1 \tag{6.14}$$

Because insurance is is costly, we finally have:

$$\beta[V_E - V_U]p'(a) < 1 \tag{6.15}$$

We know that $p''(a) < 0$, which implies that the search effort in the unemployment insurance case is higher than in autarky when agents have consumption given to them. However, agents will search less than what they should.

7 Solutions to the Incentive Problem

Suppose that agents can only observe and control consumption. By imposing the FOCs from the autarky case, we let them choose their effort level. The maximization problem is now as follows:

$$\begin{aligned} C(V) = & \min_{c,a,V_U} \{c + \beta[1 - p(a)]C(V_U)\} \\ & \text{s.t.} \\ & V \leq u(c) - a + \beta[p(a)V_E + (1 - p(a))V_U] \\ \beta p'(a)[V_E - V_u] \leq 1 & \end{aligned} \tag{7.1}$$

This formulation allows to correct the incentive problem we saw before. Let θ and λ be the multipliers for the first and second restriction respectively:

$$\frac{\partial C(V)}{\partial c} = 1 - \theta u'(c) \tag{7.2}$$

$$\frac{\partial C(V)}{\partial a} = -\beta p'(a)C(V_U)1 - \theta[1 - \beta p'(a)(V_E - V_U)] - \lambda p''(a)(V_E - V_U) \tag{7.3}$$

$$\frac{\partial C(V)}{\partial V_U} = -\beta p'(a)C'(V_U) - \theta[\beta(1 - p'(a))] + \lambda p'(a) \tag{7.4}$$

So, now we have:

$$\frac{1}{u'(c)} = \theta \tag{7.5}$$

$$C(V_U) = \theta \left[\frac{1}{\beta p'(a)} - (V_E - V_U) \right] - \lambda \frac{p''(a)}{p'(a)}(V_E - V_U) \tag{7.6}$$

$$C'(V_U) = \theta - \lambda \frac{p'(a)}{1 - p'(a)} \tag{7.7}$$

Using the envelope condition:

$$\frac{\partial C(V)}{\partial V} = \theta \tag{7.8}$$

Substituting into (7.7):

$$C'(V_U) = \frac{\partial C(V)}{\partial V} - \lambda \frac{p'(a)}{1 - p'(a)} \tag{7.9}$$

Because C is convex and $-\lambda \frac{p'(a)}{1-p'(a)} > 0$, we now have that $V_U < V$, which implies that consumption provided by the planner decreases as the spell of unemployment extends over time. This mechanism effectively controls for the incentive problem we had before, and pushes agents to increase their search effort.